# Introdução à Econometria de Cópulas

*Aplicações para sistemas de equações simultâneas onde todas as variáveis endógenas são contínuas, não negativas e com excessos de zeros*

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Definição de sistemas de equações simultâneas (caso $2 \times 2$):

$$\begin{cases} y_1 = \alpha_1 y_2 + \beta_1 X_1 + u_1 \\ y_2 = \alpha_2 y_1 + \beta_2 X_2 + u_2 \end{cases}$$

em que: $y$s são variáveis contínuas, dependentes e endógenas; $\alpha$s são parâmetros; $\beta$s são vetores linha de parâmetros; $X$s são vetores coluna de variáveis exógenas; e, $u$s são termos de erros.

$$\begin{bmatrix} 1 & -\alpha_1 \\ -\alpha_2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \beta_1 & [0] \\ [0] & \beta_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow Ay = BX + u$$

em que: $[0]$s são vetores linha de dimensões apropriadas.

Tal estrutura é chamada de "forma estrutural" do modelo, e temos problemas em estimar esse objeto com métodos tradicionais, por conta da endogeneidade.

$$Ay = BX + u \Rightarrow A^{-1}Ay = A^{-1}BX + A^{-1}u \Rightarrow y = \Gamma X + v$$

Tal estrutura é chamada de "forma reduzida" do modelo, e não temos problemas em estimar esse objeto com métodos tradicionais.

A encrenca aparece quando se está interessado em estimar $A$, e então chega-se ao "problema da identificação". O que é resolvido com variáveis instrumentais.

$$\begin{cases} y_1 = \alpha_1 y_2 + \beta_{11} x + \beta_{12} z_1 + u_1 \\ y_2 = \alpha_2 y_1 + \beta_{21} x + \beta_{22} z_2 + u_2 \end{cases}$$

$$\begin{bmatrix} 1 & -\alpha_1 \\ -\alpha_2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} & 0 \\ \beta_{21} & 0 & \beta_{22} \end{bmatrix} \begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{1-\alpha_1\alpha_2}\begin{bmatrix} 1 & \alpha_1 \\ \alpha_2 & 1 \end{bmatrix}\begin{bmatrix} \beta_{11} & \beta_{12} & 0 \\ \beta_{21} & 0 & \beta_{22} \end{bmatrix}\begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix} + \frac{1}{1-\alpha_1\alpha_2}\begin{bmatrix} 1 & \alpha_1 \\ \alpha_2 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{cases} y_1 = \dfrac{(\beta_{11}+\alpha_1\beta_{21})x + \beta_{12}z_1 + \alpha_1\beta_{22}z_2 + (u_1+\alpha_1 u_2)}{1-\alpha_1\alpha_2} \\ y_2 = \dfrac{(\alpha_2\beta_{11}+\beta_{21})x + \alpha_2\beta_{12}z_1 + \beta_{22}z_2 + (\alpha_2 u_1 + u_2)}{1-\alpha_1\alpha_2} \end{cases} \Rightarrow y = \Gamma X + v$$

$$\begin{cases} y_1 = \gamma_{11}x + \gamma_{12}z_1 + \gamma_{13}z_2 + v_1 \\ y_2 = \gamma_{21}x + \gamma_{22}z_1 + \gamma_{23}z_2 + v_2 \end{cases}$$

Note que: $\alpha_1 = \gamma_{13}/\gamma_{23}$; $\alpha_2 = \gamma_{22}/\gamma_{12}$; $\beta_{12} = \gamma_{12}(1-\alpha_1\alpha_2)$; $\beta_{22} = \gamma_{13}(1-\alpha_1\alpha_2)$; e, $\beta_{11}$ e $\beta_{21}$ são achamos por sistema residual.

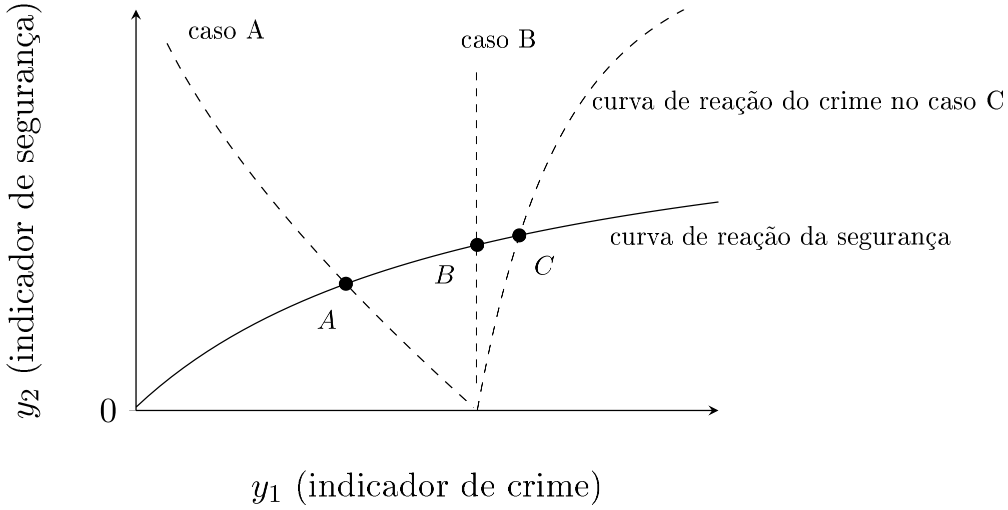
Conclusão de momento:

1. se o sistema é composto por $y$'s contínuos, e
2. se $z$'s estão disponíveis, então
3. a forma estrutural é identificada...
4. e pode ser estimada usando a forma reduzida, com estimadores de informação:
    - limitada (IV ou 2SLS), ou
    - completa (SUR, 3SLS ou FIML)
5. dentre as possibilidades, uma abordagem é a cópula
6. mas cópulas são muito mais úteis se o sistema é composto por $y$'s não contínuos, especialmente porque facilitam modelagens censuradas e hurdle

Nesse mini-curso vamos:

- Começar por um problema teórico e uma base de dados para analisá-lo
- Estimar formas reduzidas e estruturais desse problema considerando $y$'s contínuos e econometria tradicional
- Definir econometria de cópula, e então estimar formas reduzidas e estruturais desse problema considerando $y$'s contínuos e econometria de cópulas
- Estimar formas reduzidas e estruturais desse problema considerando $y$'s não contínuos e econometria de cópulas, focando em modelagens censuradas e hurdle

# Clotfelter [1978]



```
cls
clear all
import delimited "C:\Users\Usuario\Meu Drive\curso_copula\data.txt"

label variable id "city id"
label variable pop "number of inhabitants"
label variable police "police patrol officers"
label variable hom "firearm homicides"
label variable psec "private security guards"
label variable fsuicides "firearm suicides"
label variable suicides "all suicides"
label variable prison "prison industrial complex = 1"
label variable gdp "gdp per capita R$"
label variable gini "gini index 0-100"
label variable jobs "formal jobs %"
label variable unemp "unemployment %"
label variable poprural "rural population %"
label variable popmen "young men population %"

gen y1 = 100*1000*hom/pop
gen y2 = 100*psec/police

gen y1_positive = y1 > 0
gen y2_positive = y2 > 0
tab y1_positive y2_positive
su y1 if y1 > 0
su y2 if y2 > 0
```

```
y1_positiv |      y2_positive
         e |         0          1 |     Total
-----------+----------------------+----------
         0 |     1,690         33 |     1,723
         1 |     2,977        375 |     3,352
-----------+----------------------+----------
     Total |     4,667        408 |     5,075

. su y1 if y1 > 0

    Variable |        Obs        Mean    Std. dev.       Min        Max
-------------+---------------------------------------------------------
          y1 |      3,352    18.50855    11.46097   2.128747   124.2236

. su y2 if y2 > 0

    Variable |        Obs        Mean    Std. dev.       Min        Max
-------------+---------------------------------------------------------
          y2 |        408    33.95584    46.56523   .2136752        375

tw (scatter y2 y1, mcolor(gray) msize(small) msymbol(o)), ///
   ytitle(y2) xtitle(y1) xsize(5) ysize(3) ysc(r(0 375)) ylabel(0(75)375, nogrid) ///
   xsc(r(0 125)) xlabel(0(25)125) graphregion(color(white)) legend(off)
```

```
su gdp, meanonly
replace gdp = (gdp-r(min))/(r(max)-r(min))

foreach var of varlist gini jobs unemp poprural popmen {
 replace `var' = `var'/100
}

replace y1 = y1/100
replace y2 = y2/100
gen z1 = cond(suicides==0,0,fsuicides/suicides)
gen z2 = prison

ktau y1 y2 z1 z2, stats(taub p)

y1 y2 z1 z2

y1 1.0000


y2 0.1968 1.0000
0.0000

z1 0.1686 0.1192 1.0000
0.0000 0.0000

z2 -0.0387 0.2655 0.0481 1.0000
0.0012 0.0000 0.0001
```

```
global X gdp gini jobs unemp poprural popmen

sum y1 y2 z1 z2 $X

Variable Obs Mean Std. dev. Min Max

y1 5,075 .1222476 .1278994 0 1.242236
y2 5,075 .0272985 .1609926 0 3.75
z1 5,075 .1344537 .2014506 0 1
z2 5,075 .0275862 .1638002 0 1
gdp 5,075 .0346848 .0442066 0 1

gini 5,075 .4929202 .0664861 .28 .8
jobs 5,075 .4200611 .1877894 .03 .89
unemp 5,075 .0661281 .038231 .01 .42
poprural 5,075 .3860749 .2128802 .01 .98
popmen 5,075 .0923271 .0138948 .06 .46
```

```
ivregress 2sls y1 $X (y2 = z2) if y1 > 0 & y2 > 0
scalar alpha1_hat_2sls = e(b)[1,1]

ivregress 2sls y2 $X (y1 = z1) if y1 > 0 & y2 > 0
scalar alpha2_hat_2sls = e(b)[1,1]

reg3 (y1 y2 z1 $X) (y2 y1 z2 $X) if y1 > 0 & y2 > 0
scalar alpha1_hat_reg3 = e(b)[1,1]
scalar alpha2_hat_reg3 = e(b)[1,10]

sureg (y1 z1 z2 $X) (y2 z1 z2 $X) if y1 > 0 & y2 > 0
scalar alpha1_hat_sur = e(b)[1,2]/e(b)[1,11]
scalar alpha2_hat_sur = e(b)[1,10]/e(b)[1,1]

. disp alpha1_hat_2sls
-.49808317
. disp alpha1_hat_sur
-.50313148
. disp alpha1_hat_reg3
-.50313148
. disp alpha2_hat_2sls
.56024243
. disp alpha2_hat_sur
.11805669
. disp alpha2_hat_reg3
.11805669
```

```
program define fiml_gaussian
 args lnf xb1 xb2 s1 s2 r
 tempvar v1 v2 rho
  quietly {
   local y1 "$ML_y1"
   local y2 "$ML_y2"
   gen double `v1' = (`y1'-`xb1')/exp(`s1')
   gen double `v2' = (`y2'-`xb2')/exp(`s2')
   gen double `rho' = (2/_pi)*atan(`r')
   replace `lnf' = -`s1'-`s2'-ln(2*_pi)-.5*ln(1-`rho'^2)-(.5/(1-`rho'^2))*(`v1'^2-2*`rho'*`v1'*`v2'+`v2'^2)
  }
end

ml model lf fiml_gaussian (y1 y2 = z1 z2 $X) (z1 z2 $X) () () () if y1 > 0 & y2 > 0
ml check

constraint 1 _b[eq5:_cons] = 0

ml model lf fiml_gaussian (y1 y2 = z1 z2 $X) (z1 z2 $X) () () () if y1 > 0 & y2 > 0, constraint(1)
ml max, difficult iterate(50)
matrix start = e(b)

ml model lf fiml_gaussian (y1 y2 = z1 z2 $X) (z1 z2 $X) () () () if y1 > 0 & y2 > 0
ml init start
ml max, difficult iterate(50)
```

Quando o pesquisador assume normalidade, como no caso da estrutura anterior, isso pode levar a estimativas inconsistentes por má especificação.

Uma maneira simples de se lidar com isso são as cópulas. Para simplificar a notação, defina $\tilde{v}_j = v_j/\sigma_j$ onde: $v$s são erros da forma reduzida (i.e., $y_j - \Gamma_j X$); e, cada $\sigma > 0$ é um parâmetro de escala. Por construção, cada $\tilde{v}$ é média zero condicional em $X$. Então, defina $G_j$ e $g_j$ como c.d.f. e p.d.f. marginais de $\tilde{v}_j$.

Nessa configuração, a definição de cópula é: qualquer função $C$ satisfazendo $G(\tilde{v}_1, \tilde{v}_2) = C(G_1(\tilde{v}_1), G_2(\tilde{v}_2); \theta)$, em que $G$ é c.d.f. bivariada e $\theta$ é um parâmetro relacionado com uma medida de dependência das marginais

Consequentemente, $C$ gera uma distribuição conjunta usando $G_1$, $G_2$ e $\theta$.

| Distribuição | $G_1$ | $g_1$ |
|---|---|---|
| Normal | $\Phi(\tilde{v}_1)$ | $\varphi(\tilde{v}_1)$ |
| Gumbel I | $\exp\big(-\exp(-\tilde{v}_1 + \frac{\epsilon}{\sigma_1})\big)$ | $\exp\big(-\tilde{v}_1 + \frac{\epsilon}{\sigma_1} - \exp(-\tilde{v}_1 + \frac{\epsilon}{\sigma_1})\big)$ |
| Gumbel II | $1 - \exp\big(-\exp(\tilde{v}_1 - \frac{\epsilon}{\sigma_1})\big)$ | $\exp\big(\tilde{v}_1 - \frac{\epsilon}{\sigma_1} - \exp(\tilde{v}_1 - \frac{\epsilon}{\sigma_1})\big)$ |

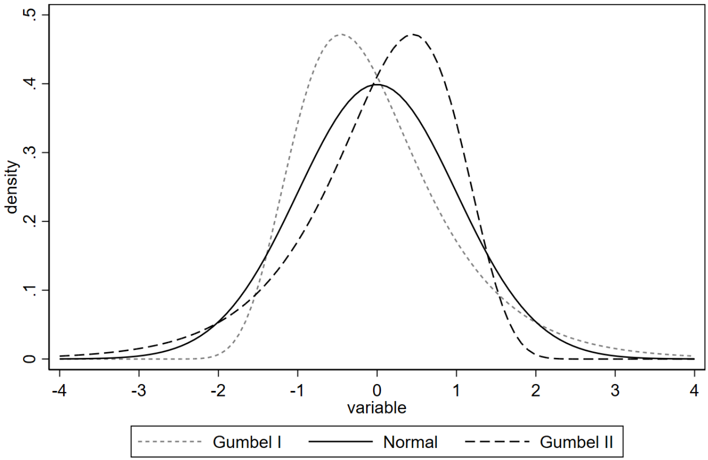
```
cls
clear all
set obs 10000

matrix g = J(sqrt(_N),1,0)
local b = 4
local t = sqrt(_N)
local i = 1
while `i' <= `t' {
 matrix g[`i',1] = -`b'+(`i'-1)/((`t'-1)/(2*`b'))
local i = `i' + 1
}
matrix g1 = g#J(`t',1,1)
matrix g2 = J(`t',1,1)#g
svmat double g1, name(x)
svmat double g2, name(y)
rename x1 v1
rename y1 v2

gen G1_gauss = normal(v1)
gen g1_gauss = normalden(v1)
gen G2_gauss = normal(v2)
gen g2_gauss = normalden(v2)
```

```
scalar b = sqrt(6)/_pi
scalar a = digamma(1)*b
gen vv1 = (v1-a)/b
gen G1_gI = exp(-exp(-vv1))
gen g1_gI = exp(-vv1-exp(-vv1))/b
gen vv2 = (v2-a)/b
gen G2_gI = exp(-exp(-vv2))
gen g2_gI = exp(-vv2-exp(-vv2))/b
replace vv1 = (v1+a)/b
replace vv2 = (v2+a)/b
gen G1_gII = 1-exp(-exp(vv1))
gen g1_gII = exp(vv1-exp(vv1))/b
gen G2_gII = 1-exp(-exp(vv2))
gen g2_gII = exp(vv2-exp(vv2))/b

tw (line g1_gI v1, sort lcolor(gray) lpattern(shortdash)) ///
   (line g1_gauss v1, sort lcolor(black) lpattern(solid)) ///
   (line g1_gII v1, sort lcolor(black) lpattern(dash)), ///
   ytitle(density) xtitle(variable) xla(-4(1)4, axis(1)) xsize(6) ysize(4) scheme(s1mono) ///
   legend(cols(3) label(1 "Gumbel I") label(2 "Normal") label(3 "Gumbel II"))
```


density
variable
Gumbel I
Normal
Gumbel II

Gaussiana:

$$C = \Phi_{12}(\Phi^{-1}G_1, \Phi^{-1}G_2; \theta)$$

$$\frac{\partial C}{\partial G_1} = \Phi\Big(\frac{\Phi^{-1}G_2 - \theta\Phi^{-1}G_1}{\sqrt{1-\theta^2}}\Big)$$

$$\frac{\partial^2 C}{\partial G_1 \partial G_2} = \frac{\varphi_{12}(\Phi^{-1}G_1, \Phi^{-1}G_2; \theta)}{\varphi(\Phi^{-1}G_1)\varphi(\Phi^{-1}G_2)}$$

Clayton:

$$C = \sqrt[-\theta]{G_1^{-\theta} + G_2^{-\theta} - 1}$$

$$\frac{\partial C}{\partial G_1} = \Big(\frac{C}{G_1}\Big)^{1+\theta}$$

$$\frac{\partial^2 C}{\partial G_1 \partial G_2} = \frac{(1+\theta)C^{1+2\theta}}{(G_1 G_2)^{1+\theta}}$$

Clayton 180:

$$C^r = G_1 + G_2 - 1 + C(1-G_1, 1-G_2)$$

$$\frac{\partial C^r}{\partial G_1} = g_1 - \Big(\frac{C(1-G_1, 1-G_2)}{1-G_1}\Big)^{1+\theta}$$

$$\frac{\partial^2 C^r}{\partial G_1 \partial G_2} = \frac{(1+\theta)\big(C(1-G_1, 1-G_2)\big)^{1+2\theta}}{\big((1-G_1)(1-G_2)\big)^{1+\theta}}$$

Clayton 270 (90):

$$C^r = G_1 - C(G_1, 1-G_2)$$

$$\frac{\partial C^r}{\partial G_1} = g_1 - \Big(\frac{C(G_1, 1-G_2)}{G_1}\Big)^{1+\theta}$$

$$\frac{\partial C^r}{\partial G_2} = \Big(\frac{C(G_1, 1-G_2)}{1-G_2}\Big)^{1+\theta}$$

$$\frac{\partial^2 C^r}{\partial G_1 \partial G_2} = \frac{(1+\theta)\big(C(G_1, 1-G_2)\big)^{1+2\theta}}{\big(G_1(1-G_2)\big)^{1+\theta}}$$

Trivedi and Zimmer [2007, p. 23]:

$$\tau = 4 \int_0^1 \int_0^1 C(G_1, G_2) dC(G_1, G_2) - 1$$

Trivedi and Zimmer [2007, p. 16]:
Normal: $\tau = 2\arcsin(\theta)/\pi$
Clayton: $\tau = \theta/(\theta + 2)$
Nelsen [2007, p. 26]:

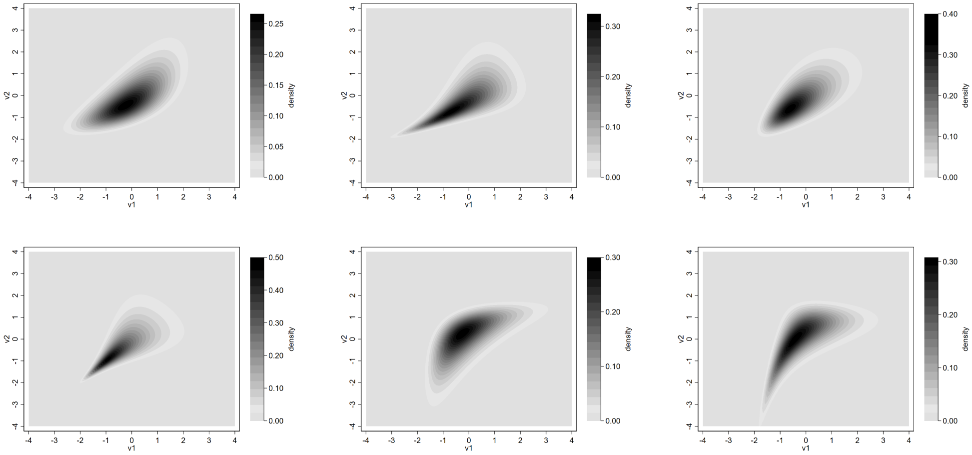
$$C^{90} = G_2 - C(1 - G_1, G_2)$$

$$C^{180} = G_1 + G_2 - 1 + C(G_1, G_2)$$

$$C^{270} = G_1 - C(G_1, 1 - G_2)$$

```
scalar theta = .707 /* tau = .5 */
gen C12 = (exp(-(invnormal(G1_gI)^2-2*theta*invnormal(G1_gI)*invnormal(G2_gauss)+invnormal(G2_gauss)^2)/(2*(1-theta^2)))/(2*_pi*sqrt(1
          (normalden(invnormal(G1_gI))*normalden(invnormal(G2_gauss)))
gen f12 = g1_gI*g2_gauss*C12
replace f12 = 0 if f12 == .
label variable f12 "density"
tw (contour f12 v1 v2, ccolors(gs14) levels(20) zlabel(#5, format(%4.2f))), xtitle(v1) ytitle(v2)  ///
   xla(-4(1)4, axis(1)) yla(-4(1)4, axis(1)) ysize(7) xsize(10) scheme(s1mono)
replace C12 = (exp(-(invnormal(G1_gI)^2-2*theta*invnormal(G1_gI)*invnormal(G2_gI)+invnormal(G2_gI)^2)/(2*(1-theta^2)))/(2*_pi*sqrt(1-t
          (normalden(invnormal(G1_gI))*normalden(invnormal(G2_gI)))
replace f12 = g1_gI*g2_gI*C12
replace f12 = 0 if f12 == .
label variable f12 "density"
tw (contour f12 v1 v2, ccolors(gs14) levels(20) zlabel(#5, format(%4.2f))), xtitle(v1) ytitle(v2)  ///
   xla(-4(1)4, axis(1)) yla(-4(1)4, axis(1)) ysize(7) xsize(10) scheme(s1mono)
replace C12 = (exp(-(invnormal(G1_gII)^2-2*theta*invnormal(G1_gII)*invnormal(G2_gI)+invnormal(G2_gI)^2)/(2*(1-theta^2)))/(2*_pi*sqrt(1
          (normalden(invnormal(G1_gII))*normalden(invnormal(G2_gI)))
replace f12 = g1_gII*g2_gI*C12
replace f12 = 0 if f12 == .
label variable f12 "density"
tw (contour f12 v1 v2, ccolors(gs14) levels(20) zlabel(#5, format(%4.2f))), xtitle(v1) ytitle(v2)  ///
   xla(-4(1)4, axis(1)) yla(-4(1)4, axis(1)) ysize(7) xsize(10) scheme(s1mono)
```

```
scalar theta = 2 /* tau = .5 */
gen C = (G1_gI^(-theta)+G2_gauss^(-theta)-1)^(-1/theta)
replace C12 = ((1+theta)*(C^(1+2*theta)))/((G1_gI*G2_gauss)^(1+theta))
replace f12 = g1_gI*g2_gauss*C12
replace f12 = 0 if f12 == .
label variable f12 "density"
tw (contour f12 v1 v2, ccolors(gs14) levels(20) zlabel(#5, format(%4.2f))), xtitle(v1) ytitle(v2)  ///
   xla(-4(1)4, axis(1)) yla(-4(1)4, axis(1)) ysize(7) xsize(10) scheme(s1mono)
replace C = (G1_gI^(-theta)+G2_gI^(-theta)-1)^(-1/theta)
replace C12 = ((1+theta)*(C^(1+2*theta)))/((G1_gI*G2_gI)^(1+theta))
replace f12 = g1_gI*g2_gI*C12
replace f12 = 0 if f12 == .
label variable f12 "density"
tw (contour f12 v1 v2, ccolors(gs14) levels(20) zlabel(#5, format(%4.2f))), xtitle(v1) ytitle(v2)  ///
   xla(-4(1)4, axis(1)) yla(-4(1)4, axis(1)) ysize(7) xsize(10) scheme(s1mono)
replace C = (G1_gII^(-theta)+G2_gI^(-theta)-1)^(-1/theta)
replace C12 = ((1+theta)*(C^(1+2*theta)))/((G1_gII*G2_gI)^(1+theta))
replace f12 = g1_gII*g2_gI*C12
replace f12 = 0 if f12 == .
label variable f12 "density"
tw (contour f12 v1 v2, ccolors(gs14) levels(20) zlabel(#5, format(%4.2f))), xtitle(v1) ytitle(v2)  ///
   xla(-4(1)4, axis(1)) yla(-4(1)4, axis(1)) ysize(7) xsize(10) scheme(s1mono)
```


v1
v2
density

```
cls
clear all
import delimited "C:\Users\Usuario\Meu Drive\curso_copula\data.txt"

label variable id "city id"
label variable pop "number of inhabitants"
label variable police "police patrol officers"
label variable hom "firearm homicides"
label variable psec "private security guards"
label variable fsuicides "firearm suicides"
label variable suicides "all suicides"
label variable prison "prison industrial complex = 1"
label variable gdp "gdp per capita R$"
label variable gini "gini index 0-100"
label variable jobs "formal jobs %"
label variable unemp "unemployment %"
label variable poprural "rural population %"
label variable popmen "young men population %"

su gdp, meanonly
replace gdp = (gdp-r(min))/(r(max)-r(min))

foreach var of varlist gini jobs unemp poprural popmen {
 replace `var' = `var'/100
}

gen y1 = 1000*hom/pop
gen y2 = psec/police
gen z1 = cond(suicides==0,0,fsuicides/suicides)
gen z2 = prison

global X gdp gini jobs unemp poprural popmen
```

```
program define fiml_gaussian
 args lnf xb1 xb2 s1 s2 r
 tempvar v1 v2 rho
  quietly {
   local y1 "$ML_y1"
   local y2 "$ML_y2"
   gen double `v1' = (`y1'-`xb1')/exp(`s1')
   gen double `v2' = (`y2'-`xb2')/exp(`s2')
   gen double `rho' = (2/_pi)*atan(`r')
   replace `lnf' = -`s1'-`s2'-ln(2*_pi)-.5*ln(1-`rho'^2)-(.5/(1-`rho'^2))*(`v1'^2-2*`rho'*`v1'*`v2'+`v2'^2)
  }
end

ml model lf fiml_gaussian (y1 y2 = z1 z2 $X) (z1 z2 $X) () () () if y1 > 0 & y2 > 0
ml check

constraint 1 _b[eq5:_cons] = 0

ml model lf fiml_gaussian (y1 y2 = z1 z2 $X) (z1 z2 $X) () () () if y1 > 0 & y2 > 0, constraint(1)
ml max, difficult iterate(50)
matrix start = e(b)

ml model lf fiml_gaussian (y1 y2 = z1 z2 $X) (z1 z2 $X) () () () if y1 > 0 & y2 > 0
ml init start
ml max, difficult iterate(50)
```

```
program define fiml_gaussian
 args lnf xb1 xb2 s1 s2 r
 tempvar v1 v2 rho
  quietly {
   local y1 "$ML_y1"
   local y2 "$ML_y2"
   gen double `v1' = (`y1'-`xb1')/exp(`s1')
   gen double `v2' = (`y2'-`xb2')/exp(`s2')
   gen double `rho' = (2/_pi)*atan(`r')
   replace `lnf' = -`s1'-`s2'-ln(2*_pi)-.5*ln(1-`rho'^2)-(.5/(1-`rho'^2))*(`v1'^2-2*`rho'*`v1'*`v2'+`v2'^2)
  }
end

ml model lf fiml_gaussian (y1 y2 = z1 z2 $X) (z1 z2 $X) () () () if y1 > 0 & y2 > 0
ml check

constraint 1 _b[eq5:_cons] = 0

ml model lf fiml_gaussian (y1 y2 = z1 z2 $X) (z1 z2 $X) () () () if y1 > 0 & y2 > 0, constraint(1)
ml max, difficult iterate(50)
matrix start = e(b)

ml model lf fiml_gaussian (y1 y2 = z1 z2 $X) (z1 z2 $X) () () () if y1 > 0 & y2 > 0
ml init start
ml max, difficult iterate(50)
estimates store gauss_n_n
```

```
program define fiml_clayton
 args lnf xb1 xb2 s1 s2 r
 tempvar v1 v2 theta G1 g1 G2 g2 D C12
  quietly {
   local y1 "$ML_y1"
   local y2 "$ML_y2"
   gen double `v1' = (`y1'-`xb1')/exp(`s1')
   gen double `v2' = (`y2'-`xb2')/exp(`s2')
   gen double `theta' = exp(`r')
   gen double `G1' = exp(-exp(-`v1'))
   gen double `g1' = exp(-`v1'-exp(-`v1'))
   gen double `G2' = exp(-exp(-`v2'))
   gen double `g2' = exp(-`v2'-exp(-`v2'))
   gen double `D' = (`G1'^(-`theta')+`G2'^(-`theta')-1)^(-1/`theta')
   gen double `C12' = ((1+`theta')/((`G1'*`G2')^(1+`theta')))*(`D'^(1+2*`theta'))
   replace `lnf' = ln(max(1e-20,(`g1'/exp(`s1'))*(`g2'/exp(`s2'))*`C12'))
  }
end

ml model lf fiml_clayton (y1 y2 = z1 z2 $X) (z1 z2 $X) () () () if y1 > 0 & y2 > 0
ml check

constraint 1 _b[eq5:_cons] = 0

ml model lf fiml_clayton (y1 y2 = z1 z2 $X) (z1 z2 $X) () () () if y1 > 0 & y2 > 0, constraint(1)
ml max, difficult iterate(50)
matrix start = e(b)

ml model lf fiml_clayton (y1 y2 = z1 z2 $X) (z1 z2 $X) () () () if y1 > 0 & y2 > 0
ml init start
ml max, difficult iterate(50)
estimates store clayton_gI_gI
```

```
esttab gauss_n_n clayton_gI_gI, se stats(ll)

(1) (2)
y1 y1

eq1
z1 0.0162 0.0365
(0.0463) (0.0349)

z2 -0.0279 -0.0596***
(0.0187) (0.0134)

****

eq2
z1 0.00191 0.0405
(0.156) (0.0831)

z2 0.0556 0.0883**
(0.0632) (0.0333)

***

ll 97.26 294.2
```

$$y_j = \begin{cases} y_j^* & \text{se } \min\{y_j^*, y_j^\bullet\} > 0 \\ 0 & \text{c.c.} \end{cases}$$

em que: $y_j^* = \gamma_j y_{-j} + \alpha_j' X_j + u_j$; $y_j^\bullet = \delta_j' \mathcal{X}_j + v_j$.
Censura: $y_j^* = y_j^\bullet$.

| | many $y_2 = 0$ | few $y_2 = 0$ |
|---|---|---|
| many $y_1 = 0$ | $\begin{cases} y_1^* = \gamma_1 y_2 + \alpha_1' X_1 + u_1 \\ y_2^* = \gamma_2 y_1 + \alpha_2' X_2 + u_2 \end{cases}$ | $\begin{cases} y_1^* = \gamma_1 y_2 + \alpha_1' X_1 + u_1 \\ y_2 = \gamma_2 y_1 + \alpha_2' X_2 + u_2 \end{cases}$ |
| few $y_1 = 0$ | $\begin{cases} y_1 = \gamma_1 y_2 + \alpha_1' X_1 + u_1 \\ y_2^* = \gamma_2 y_1 + \alpha_2' X_2 + u_2 \end{cases}$ | $\begin{cases} y_1 = \gamma_1 y_2 + \alpha_1' X_1 + u_1 \\ y_2 = \gamma_2 y_1 + \alpha_2' X_2 + u_2 \end{cases}$ |

Tabela: Structural forms to an empirical study based on Amemiya [1974], where: $\gamma$s are scalar-valued parameters; $\alpha$s and $X$s are column vectors of parameters and controls (including constants), respectively; and, $u$s are error terms zero mean conditional on controls.

| | $y_2 = 0$ | $y_2 > 0$ |
|---|---|---|
| $y_1 = 0$ | $\begin{cases} y_1^* = \alpha_1' X_1 + u_1 \\ y_2^* = \alpha_2' X_2 + u_2 \end{cases}$ | $\begin{cases} y_1^* = \delta_1' X + (u_1 + \gamma_1 u_2) \\ y_2^* = \alpha_2' X_2 + u_2 \end{cases}$ |
| $y_1 > 0$ | $\begin{cases} y_1^* = \alpha_1' X_1 + u_1 \\ y_2^* = \delta_2' X + (u_2 + \gamma_2 u_1) \end{cases}$ | $\begin{cases} y_1^* = \big(\delta_1' X + (u_1 + \gamma_1 u_2)\big)/(1 - \gamma_1\gamma_2) \\ y_2^* = \big(\delta_2' X + (u_2 + \gamma_2 u_1)\big)/(1 - \gamma_1\gamma_2) \end{cases}$ |

Tabela: Reduced forms of Amemiya's design, where: $X_1 = (z_1, X_{12}')'$ and $X_2 = (z_2, X_{12}')'$, being $z$s instruments to identify the structural form and $X_{12}$ common to both equations; $X = (z_1, z_2, X_{12}')'$; $\alpha_1 = (\alpha_{11}, \alpha_{12}')'$ and $\alpha_2 = (\alpha_{21}, \alpha_{22}')'$; $\delta_1 = (\alpha_{11}, \gamma_1\alpha_{22}, \alpha_{12}' + \gamma_1\alpha_{21}')'$ and $\delta_2 = (\gamma_2\alpha_{11}, \alpha_{22}, \gamma_2\alpha_{12}' + \alpha_{21}')'$.

These four reduced forms return to structural form only when $\gamma_1\gamma_2 < 1$. To illustrate this, consider the two reduced forms for the case $y_1 > 0$: $y_2 = 0$ implies in $y_2^* = \delta_2'X + u_2 + \gamma_2 u_1 < 0$; and $y_2 > 0$ implies in $y_2^* = (\delta_2'X + u_2 + \gamma_2 u_1)/(1 - \gamma_1\gamma_2) > 0$. Therefore, $\gamma_1\gamma_2 < 1$ must occur for the two sets to be mutually exclusive. Maddala [1983, Section 7.5] calls this of "logical-consistency condition".

Once the logical-consistency condition is ensured, the researcher needs exactly one instrument in each equation to identify the entire Amemiya's design, as in any regular simultaneous equations model. Given this, we can focus on the following specification:

$$\begin{cases} y_1^* = \beta_1'X + v_1 \\ y_2^* = \beta_2'X + v_2 \end{cases}$$

where: $\beta_1 = (\beta_{11}, \beta_{12}, \beta_{13}')$ and $\beta_2 = (\beta_{21}, \beta_{22}, \beta_{23}')$; and, $v_1 = (u_1 + \gamma_1 u_2)/(1 - \gamma_1\gamma_2)$ and $v_2 = (u_2 + \gamma_2 u_1)/(1 - \gamma_1\gamma_2)$.

In this setup, the structural form is exactly identified by: $\gamma_1 = \beta_{12}/\beta_{22}$, $\gamma_2 = \beta_{21}/\beta_{11}$, $\alpha_{11} = (1 - \gamma_1\gamma_2)\beta_{11}$, $\alpha_{22} = (1 - \gamma_1\gamma_2)\beta_{22}$, and $\alpha_{12} = \beta_{13} - \gamma_1\beta_{23}$, $\alpha_{21} = \beta_{23} - \gamma_2\beta_{13}$.

Finally, the likelihood function is:

$$L = \prod_{(y_1=0, y_2=0)} F_{12}(y_1 = 0, y_2 = 0) \prod_{(y_1>0, y_2=0)} f_1(y_1) \times F_2(y_2 = 0 \mid y_1) \prod_{(y_1=0, y_2>0)} f_2(y_2) \times F_1(y_1 = 0 \mid y_2) \prod_{(y_1>0, y_2>0)} f_{12}(y_1, y_2)$$

where: $F_{12}$, $f_{12}$, $f_1$ and $f_2$ represent joint c.d.f., joint p.d.f., and marginals p.d.f. of $y_1^*$ and $y_2^*$, respectively; $F_1$ and $F_2$ are conditional probabilities of $y_1^*$ and $y_2^*$; and, all is conditional on $X$.

| Term | | | | Result to substitute |
|---|---|---|---|---|
| $F_{12}$ | $=$ | $C$ | $=$ | $C\Big(G_1(\tilde{v}_1), G_2(\tilde{v}_2); \theta\Big)$ |
| $f_1 \times F_2 \ ; \ f_2 \times F_1$ | $=$ | $\dfrac{\partial C}{\partial y_1} \ ; \ \dfrac{\partial C}{\partial y_2}$ | $=$ | $\dfrac{g_1}{\sigma_1} \times \dfrac{\partial C}{\partial G_1} \ ; \ \dfrac{g_2}{\sigma_2} \times \dfrac{\partial C}{\partial G_2}$ |
| $f_{12}$ | $=$ | $\dfrac{\partial^2 C}{\partial y_1 \partial y_2}$ | $=$ | $\dfrac{g_1}{\sigma_1} \times \dfrac{g_2}{\sigma_2} \times c$ |

Tabela: Terms of the likelihood function in a copula approach.

```
tobit y1 $X z1 z2, ll(0)
matrix start1 = (e(b)[1..1,1..6],e(b)[1,9],e(b)[1,7],e(b)[1,8])
scalar sd1 = ln(sqrt(e(b)[1,10]))
tobit y2 $X z1 z2, ll(0)
matrix start2 = (e(b)[1..1,1..6],e(b)[1,9],e(b)[1,7],e(b)[1,8])
scalar sd2 = ln(sqrt(e(b)[1,10]))

*define start values:
matrix start = (start1,sd1,start2,sd2,0)
matrix coleq start = eq1 eq1 eq1 eq1 eq1 eq1 eq1 eq2 eq3 eq4 eq5 eq5 eq5 eq5 eq5 eq5 eq5 eq6 eq7 eq8 eq9
matrix colnames start = gdp gini jobs unemp poprural popmen _cons _cons _cons _cons gdp gini jobs unemp poprural popmen _cons _cons _c
matrix list start

*define copula:
program define copula
 version 17.0
 args lf xb1 b11 b12 s1 xb2 b21 b22 s2 r
 tempvar sigma1 sigma2 theta beta11 beta12 beta21 beta22 v1 v2 G1 G2 g1 g2 D C C1 C2 C12 f1P f2P f12
  quietly {
   local y1 "$ML_y1"
   local y2 "$ML_y2"
-> ESSE PROGRAMA É BEM GRANDE <-
   tempname minv
   scalar `minv' = 1e-20
   replace `lf' = ln(min(1-`minv',max(`minv',`C'))) if `y1' == 0 & `y2' == 0
   replace `lf' = ln(max(`minv',`f1P')) if `y1' > 0 & `y2' == 0
   replace `lf' = ln(max(`minv',`f2P')) if `y1' == 0 & `y2' > 0
   replace `lf' = ln(max(`minv',`f12')) if `y1' > 0 & `y2' > 0
   }
end
```

```
*i and j define marginals, and k defines copula:
gen i = 1
gen j = 1
gen k = 1

/*

*checking:
ml model lf copula (y1 y2 z1 z2 i j k = $X) () () () ($X) () () () ()
ml check
replace k = 2
ml model lf copula (y1 y2 z1 z2 i j k = $X) () () () ($X) () () () ()
ml check
replace k = 3
ml model lf copula (y1 y2 z1 z2 i j k = $X) () () () ($X) () () () ()
ml check
replace k = 1

*/

constraint 1 _b[eq9:_cons] = 0

/*

*checking (2 tobit = these results):
ml model lf copula (y1 y2 z1 z2 i j k = $X) () () () ($X) () () () (), constraint(1) technique(nr 25 bhhh 5 dfp 5 bfgs 5 nr 10)
ml init start
ml max, difficult iterate(50) tolerance(1e-5) ltolerance(1e-5) nrtolerance(1e-5) skip

ml model lf copula (y1 y2 z1 z2 i j k = $X) () () () ($X) () () () (), technique(nr 25 bhhh 5 dfp 5 bfgs 5 nr 10)
ml init start
ml max, difficult iterate(50) tolerance(1e-5) ltolerance(1e-5) nrtolerance(1e-5) skip
matrix start1 = e(b)
```

```
*procedure:
tic
local kk = 1
 while ‘kk’ <= 3 {
 replace k = ‘kk’
 matrix L = J(3,3,0)
 matrix A = J(3,3,0)
 matrix B11 = J(3,3,0)
 matrix B12 = J(3,3,0)
 matrix B21 = J(3,3,0)
 matrix B22 = J(3,3,0)
 matrix G1 = J(3,3,0)
 matrix G2 = J(3,3,0)
 matrix GG = J(3,3,0)
 matrix T = J(3,3,0)
  local ii = 1
   while ‘ii’ <= 3 {
   replace i = ‘ii’
    local jj = 1
    while ‘jj’ <= 3 {
    replace j = ‘jj’
     ml model lf copula (y1 y2 z1 z2 i j k = $X) () () () ($X) () () () (), technique(nr 20 bhhh 10 dfp 5 bfgs 5 nr 5 bhhh 5)
     ml init start‘kk’
     ml max, difficult iterate(50) tolerance(1e-3) ltolerance(1e-3) nrtolerance(1e-3) skip
     estimates store copula_‘ii’‘jj’_‘kk’
     matrix L[‘ii’,‘jj’] = e(ll)
     matrix A[‘ii’,‘jj’] = 2*(e(k)-e(ll))
     matrix B11[‘ii’,‘jj’] = _b[eq2:_cons]
     matrix B12[‘ii’,‘jj’] = _b[eq3:_cons]
     matrix B21[‘ii’,‘jj’] = _b[eq6:_cons]
     matrix B22[‘ii’,‘jj’] = _b[eq7:_cons]
     matrix G1[‘ii’,‘jj’] = B12[‘ii’,‘jj’]/B22[‘ii’,‘jj’]
     matrix G2[‘ii’,‘jj’] = B21[‘ii’,‘jj’]/B11[‘ii’,‘jj’]
```

```
replace i = 1
replace j = 2
replace k = 2
ml model lf copula (y1 y2 z1 z2 i j k = $X) () () () ($X) () () () (), technique(nr 20 bhhh 10 dfp 5 bfgs 5 nr 5 bhhh 5)
ml init start2
ml max, difficult iterate(50) tolerance(1e-5) ltolerance(1e-5) nrtolerance(1e-5) skip
estimates store copula_best
matrix reduced = e(b)

*structural estimates:
scalar gamma1 = _b[eq3:_cons]/_b[eq7:_cons]
scalar gamma2 = _b[eq6:_cons]/_b[eq2:_cons]
scalar a11 = (1-gamma1*gamma2)*_b[eq2:_cons]
scalar a22 = (1-gamma1*gamma2)*_b[eq7:_cons]
scalar a1_cons = _b[eq1:_cons]-gamma1*_b[eq5:_cons]
scalar a1gdp = _b[eq1:gdp]-gamma1*_b[eq5:gdp]
scalar a1gini = _b[eq1:gini]-gamma1*_b[eq5:gini]
scalar a1jobs = _b[eq1:jobs]-gamma1*_b[eq5:jobs]
scalar a1unemp = _b[eq1:unemp]-gamma1*_b[eq5:unemp]
scalar a1poprural = _b[eq1:poprural]-gamma1*_b[eq5:poprural]
scalar a1popmen = _b[eq1:popmen]-gamma1*_b[eq5:popmen]
scalar a2_cons = _b[eq5:_cons]-gamma2*_b[eq1:_cons]
scalar a2gdp = _b[eq5:gdp]-gamma2*_b[eq1:gdp]
scalar a2gini = _b[eq5:gini]-gamma2*_b[eq1:gini]
scalar a2jobs = _b[eq5:jobs]-gamma2*_b[eq1:jobs]
scalar a2unemp = _b[eq5:unemp]-gamma2*_b[eq1:unemp]
scalar a2poprural = _b[eq5:poprural]-gamma2*_b[eq1:poprural]
scalar a2popmen = _b[eq5:popmen]-gamma2*_b[eq1:popmen]
scalar sigma1 = exp(_b[eq4:_cons])
scalar sigma2 = exp(_b[eq8:_cons])*(sqrt(6)/_pi)
scalar tau = exp(_b[eq9:_cons])/(exp(_b[eq9:_cons])+2)

*delta method:
```

```
matrix list reduced

matrix b1 = (reduced[1,8..9],reduced[1,1..7])'
matrix b2 = (reduced[1,18..19],reduced[1,11..17])'
gen cte = 1
mkmat z1 z2 gdp gini jobs unemp poprural popmen cte, matrix(X)
matrix xb1 = X*b1
matrix xb2 = X*b2
svmat double xb1, name(y1star)
svmat double xb2, name(y2star)
rename y1star1 y1star
rename y2star1 y2star
replace y2star = y2star-digamma(1)

tw (scatter y2star y1star if y1star > -.2 & y1 < .8 & y2star > -1.5 & y2 < 2, mcolor(black) msize(tiny) msymbol(x)) ///
   (scatter y2 y1 if y1star > -.2 & y1 < .8 & y2star > -1.5 & y2 < 2, mcolor(gray) msize(small) msymbol(dh)), ///
   ytitle(y2*) xtitle(y1*) xsize(10) ysize(6) ysc(r(-1.5 2)) ylabel(-1.5(.5)2, nogrid) ///
   xsc(r(-.2 .8)) xlabel(-.2(.2).8) graphregion(color(white)) legend(off)
```

Deb et al. [2014] + Cheng and Long [2018]

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